Thanks to Dr. Tarek Sayed Ahmed whose recent work was a motivation of the subject of this paper.

Let L denote a non-empty countable relational language (this entails no loss of generality): $L = (R_i)_{i\in I}$ where I is a non-empty countable index set and R_i is an n_i-ary relation symbol. Denote by X_L the space

$$X_L = \prod_{i\in I} 2^{(\mathbb{N}^{n_i})}.$$

We view the space X_L as the space of countably infinite *L-structures*.

A *fragment* F of $L_{\omega_1\omega}$ is a set of formulas in $L_{\omega_1\omega}$ containing all atomic formulas, closed under subformulas, negation, quantifiers and finite conjunctions and disjunctions.

Definition 0.1. For $\varphi(=\varphi(\bar{v}))$ a formula of $L_{\omega_1\omega}$ and $\bar{s}$ a finite sequence from $\mathbb{N}$ of appropriate length (i.e, $\bar{s} \in {}^{|\Delta\varphi|}\omega$), let

$$Mod(\varphi, \bar{s}) = \{x \in X_L : \mathcal{U}_x \models \varphi[\bar{s}]\},$$

where $\varphi[\bar{s}]$ denotes the sentence obtained from the formula $\varphi(\bar{v})$ by substituting $\bar{s}$ for the free variables. (If φ is a sentence, we write $Mod(\varphi)$ for $Mod(\varphi, ())$.).

Let t_F be the topology on X_L generated by $\mathcal{B}_F = \{Mod(\varphi, \bar{s}) : \varphi \in F, \bar{s} \in {}^{|\Delta\varphi|}\omega\}$. By a result of Sami (See [4]), t_F is a Polish topology on X_L.

Let F be a fragment of $L_{\omega_1\omega}$. We say that $x, y \in X_L$ (or *their corresponding structures*) are *separable in* F, if there is $\varphi \in F$ such that $|\varphi^x| \neq |\varphi^y|$, where $\varphi^x = \{\bar{s} \in {}^{|\Delta\varphi|}\omega : \mathcal{U}_x \models \varphi[\bar{s}]\}$. (It is clear that if two structures are separable in some fragment, then they are non-isomorphic). Notice that, if φ is a sentence, then for all x, either φ^x is empty or else contains only the empty sequence.

For F a fragment

$$E_F = \{(x, y) \in X_L \times X_L : \text{ For all } \varphi \in F,\ |\varphi^x| = |\varphi^y|\}.$$

Theorem 0.2. *For F a countable fragment of $L_{\omega_1\omega}$, E_F is Borel in the product topology $(X_L, t_F) \times (X_L, t_F)$.*

For every $\varphi \in F$ which is not a sentence, select a bijection $\mu_\varphi : \mathbb{N} \longrightarrow {}^{|\Delta\varphi|}\omega$. If φ is a sentence, let μ_φ be the constant map from $\mathbb{N}$ to $\mathbb{N}$ that sends everything to 1 (a value that cannot be the empty sequence). It is clear that, for a set $X \subseteq {}^{|\Delta\varphi|}\omega$,

$$X \text{ is infinite } \textit{iff } (\forall n)(\exists m > n)\mu_\varphi(m) \in X.$$

$$
\begin{aligned}
|X| = |Y| \in \omega \Longleftrightarrow & (\exists n)(\exists f, g \in Inj(n, {}^{|\Delta\varphi|}\omega))(f^*(n) = X \wedge g^*(n) = Y) \\
& \Longrightarrow (\exists n)(\exists f, g \in Inj(n, {}^{|\Delta\varphi|}\omega))(f^*(g^{-1}(Y)) = X \wedge \\
& g^*(f^{-1}(X)) = Y) \\
& \Longrightarrow (\exists n)(\exists f, g \in Inj(n, {}^{|\Delta\varphi|}\omega))(X \subseteq f^*(n) \wedge Y \subseteq g^*(n) \\
& \wedge g^{-1}(Y) = f^{-1}(X)) \\
& \Longrightarrow (\exists n)(\exists f, g \in Inj(n, {}^{|\Delta\varphi|}\omega))(|X| = |f^{-1}(X)| \wedge |Y| = \\
& |g^{-1}(Y)| \wedge |g^{-1}(Y)| = |f^{-1}(X)|) \\
& \Longrightarrow |X| = |Y| \in \omega.
\end{aligned}
$$

$$
f^*(g^{-1}(Y)) = X \Longleftrightarrow (\forall t)[t \in X \Leftrightarrow g(f^{-1}(t)) \in Y].
$$

Now we are ready to prove our main theorem.

Corollary 0.3. *Let T be a first order theory in a countable language. If T has an uncountable set of pairwise separable (in any countable fragment of $L_{\omega_1\omega}$) countable models, then it has such a set of size $2^{\aleph_0}$ (and so has $2^{\aleph_0}$ non-isomorphic countable models).*

The above corallary can have other versions. We can talk about any set of models of T whose corresponding set of codes is G_δ in X_L. For example, suppose we are given a certain countable family, $\{\Gamma_i : i < \omega\}$, of non-isolated n-types ($n \in \omega$) of T (see [3]).

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